Advanced Topics in Applied Probability
- Introduction to Lattice Models

Exercises denoted by ($\star$) are harder or use additional theory.

Exercises – Set 3

1. (**Quasi-inverse**) Let $\phi\colon X \to Y$ be a map between two metric spaces (X, d_X) and (Y, d_Y). We say that $\psi\colon Y \to X$ is a *quasi-inverse* of ϕ if there exists $C \in (0, \infty)$ such that

$$d_X\big((\psi \circ \phi)(x), x\big) \leq C \quad \text{for all } x \in X,$$
$$d_Y\big((\phi \circ \psi)(y), y\big) \leq C \quad \text{for all } y \in Y.$$

   (a) Suppose $\phi\colon X \to Y$ is a quasi-isometry. Show that it has a quasi-inverse, which is a quasi-isometry as well.

   (b) We say that (X, d_X) and (Y, d_Y) are quasi-isometric if there exists a quasi-isometry between them. Show that for metric spaces, being quasi-isometric is an equivalence relation.

2. (**Quasi-isometric trees**) Let T and T' be two infinite trees of bounded vertex degree, and all of whose degrees are at least three. Prove that T and T' are quasi-isometric.

3. ($\star$) (**Recurrence/transience criterion**) Let $f\colon \{0, 1, 2, \ldots\} \to \{0, 1, 2, \ldots\}$ be a function such that

$$f(n) \leq f(n+1) \leq f(n) + 1 \qquad \text{for each } n \geq 0.$$

   Show that the SRW starting from 0 on the subgraph of $\mathbb{Z}^{d+1}$ induced by the vertex set

$$V := \{(x, y_1, y_2, \ldots, y_d) \in \mathbb{Z}^{d+1} \,|\, x \geq 0,\ |y_j| \leq f(x) \text{ for all } j = 1, 2, \ldots, d\}$$

   is transient if and only if

$$\sum_{n=1}^{\infty} (1 + f(n))^{-d} < \infty.$$

   [Hint: for direction "$\Leftarrow$", one can e.g. consider certain random paths from the origin to infinity built from coordinates of type x and $y_j = \lfloor U_j f(x) \rfloor$ for $j = 1, 2, \ldots, d$, where U_j are i.i.d. uniform random variables on $[0, 1]$.]

4. (**Recurrence/transience is quasi-isometry invariant**) Let $G = (V, E)$ and $G' = (V', E')$ be two countably infinite graphs and $\phi\colon V \to V'$ a quasi-isometry. Let $0 \in V$ be a chosen "origin" vertex and let j be a $0/\infty$ flow such that $|j| = 1$ and $\mathcal{E}(j) < \infty$. Define for each oriented edge $\langle a, b\rangle \in E'$ the flow

$$j'_{a,b} := \sum_{\substack{\langle u,v\rangle \in E \\ \text{(oriented)}}} j_{u,v}\, \mathbf{1}\{\langle a, b\rangle \in \phi(\langle u, v\rangle)\}$$

   where all edges are considered as oriented and for each $\langle u, v\rangle \in E$, we take $\phi(\langle u, v\rangle)$ to be a chosen oriented path from $\phi(u)$ to $\phi(v)$ on G' which has minimal length (note that there might be many choices and we pick one once and for all). Show that j' is a $\phi(0)/\infty$ flow such that $|j'| = 1$ and $\mathcal{E}(j') < \infty$.

5. (**Cylinder sets for UST**) [The purpose of this problem is to formalize what the weak limit of USTs on $\mathbb{Z}^d$ means.]

Consider the lattice $\mathbb{Z}^d$ regarded as a graph (V,E) and with $d \geq 2$. Let $\Omega = \{0,1\}^E$ be endowed with the *cylinder sigma-algebra* $\mathcal{F}$ generated by the *cylinder sets*

$$\big\{ \big(\omega(e)\big)_{e\in E} \,\big|\, \omega(e_1) = \varepsilon_1 \,, \omega(e_2) = \varepsilon_2, \, \ldots, \, \omega(e_k) = \varepsilon_k \big\} \tag{1}$$

for integers $k \in \mathbb{N}$, distinct edges $e_1, e_2, \ldots, e_k \in E$, and Boolean numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \in \{0,1\}$.

Similarly, let Ω be endowed with the (Tychonoff) *product topology*, in which the cylinder sets (1) form a basis of all open sets. Note that the product topology is strictly coarser than the *box topology*, in which a basis of all open sets would be given by Cartesian products of open sets.

For each fixed $n \in \mathbb{N}$, consider the UST probability measure $\mathbb{P}_n$ on $Q_n := [-n,n]^d \cap \mathbb{Z}^d$ regarded as a graph (V_n, E_n), that is, the uniform measure on the set of spanning trees of this graph. View $\mathbb{P}_n$ alternatively as a probability measure on $(\Omega, \mathcal{F})$, denoted μ_n, by viewing the UST $T_n = (T_n(e))_{e\in E}$ as a random element of Ω with $T_n(e) = 1$ if and only if $e \in T_n$. In particular, all elements in Ω which have a nonzero coordinate at an edge $e \notin E_n$ have μ_n-probability zero, as also do all elements in Ω which would result in a graph which is not a spanning tree of (V_n, E_n).

In this way, we have a sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$ on $(\Omega, \mathcal{F})$.

(ü) Recall why the box topology doesn't work for the below statements.

(a) ($\star$) Recall e.g. from your basic probability course (or Ch.2.3 in Grimmett's book) the notion of *weak convergence* of measures.

(b) ($\star$) Recall that on $(\Omega, \mathcal{F})$ with $\mathcal{F}$ the cylinder sigma-algebra, we have that μ_n converge weakly to μ as $n \to \infty$ if and only if

$$\lim_{n\to\infty} \mu_n[C] = \mu[C] \qquad \text{for all cylinder sets } C \in \mathcal{F}. \tag{2}$$

(c) ($\star$) Recall that for $(\Omega, \mathcal{F})$ as above, if the limit (2) exists, then it defines a measure μ on $(\Omega, \mathcal{F})$, and furthermore, this limit μ is a probability measure.

(d) Phrase the cylinder set (1) in terms of an event for the UST $T_n = (T_n(e))_{e\in E}$ on Q_n. Then show that all such "cylinder events" can be expressed as linear combinations of events of the form $\{e_1, e_2, \ldots, e_k \in T_n\}$. [Hint: Inclusion/exclusion to express events like $\{e \notin T_n\}$.]

(e) Recalling from the lecture that we know already that $\lim_{n\to\infty} \mu_n[e_1 \in T_n]$ exists for one edge, show that for each $k \in \mathbb{N}$, the limit

$$\lim_{n\to\infty} \mu_n[e_1, e_2, \ldots, e_k \in T_n]$$

exists, and conclude that the limit (2) exists for the UST measures μ_n.

[Hint: Recall Rayleigh's principle and the proof ingredients for one edge.]

(f) ($\star$) Fix vertices $u, v \in V$ and consider the event

$$\{u \text{ is connected to } v\} := \bigcup_{\substack{\gamma \text{ lattice path} \\ \text{from } u \text{ to } v}} \big\{ \big(\omega(e)\big)_{e\in E} \,\big|\, \omega(e) = 1 \text{ for all egdes } e \in \gamma \big\}.$$

What is the probability of such an event for $T_n \sim \mu_n$? What can happen in the limit $n \to \infty$?

Upon finding mistakes and/or typos, please contact me!