Around the conformal anomaly

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The *conformal restriction property* of Schramm-Loewner evolution (SLE) type measures has been a key feature — while still somewhat mysterious — of many developments involving SLE/CLE problems and their relationship to discrete lattice paths. For example, part of the conjecture that the self-avoiding walk (SAW) should converge in the scaling limit to an SLE(8/3) path is based on the exact conformal restriction invariance of SAW measures [1]. Generalizations of SLE type curves on Riemann surfaces, i.e., natural measures on paths and loops enjoying the symmetries underlying critical models and conformal field theory (CFT), were proposed by Kontsevich [2] and Werner [3] building on their known conformal restriction property [4] and on somewhat analogous conformal Markov properties of the discrete models (especially of their phase boundaries). Geometrically, an infinitesimal version of the conformal restriction property relates the loop measures to Malliavin's proposal [5] for a canonical diffusion on the diffeomorphism group $\mathrm{Diff}(S^1)$ of the circle. These measures, also termed "*MKS measures*" were soon thereafter investigated by a few people, involving Kontsevich & Suhov [6] Friedrich [7], and Dubédat [8, 9], among others. It is believed that the conformal restriction covariance property uniquely determines the MKS loop measures [6].

Pertaining to the general picture, the conformal restriction property can be formulated precisely in terms of an "*anomaly*" that describes the response of the system to deformations of the underlying space (domain, complex structure, etc.) For example, if μ_D is an SLE(κ) measure on a domain D and μ_U is an SLE(κ) measure on its subdomain $U \subset D$, then the two measures compare as

$$\frac{\mathrm{d}\mu_U}{\mathrm{d}\mu_D}(\gamma) = 1\{\gamma \subset U\} \; \exp\left(\tfrac{1}{2}c(\kappa)\, m_D(\gamma, D \setminus U)\right)$$

where $m_D(\gamma, D \setminus U)$ is a conformal invariant and $c(\kappa) = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$ is a function of the diffusivity parameter $\kappa \in (0, 4]$, encoding the strength of the anomaly (being zero for $\kappa = 8/3$). It is well known [4, 10, 11] that for chordal or loop SLE(κ), the factor $m_D(\gamma, D \setminus U)$ equals the mass of Brownian loops in D intersecting both γ and $D \setminus U$. (It also has other alternative expressions, e.g., in terms of Schwarzian derivatives of uniformizing maps [4], relating it closer to infinitesimal deformations.) Brownian loop measure being infinite in some situations (though renormalizable), it may be preferable to seek other presentations of the anomaly.

Sid Maibach and I have investigated the universality of this anomaly [12], in the spirit of Kontsevich's geometric framework [2, 6], also motivated by the point of view of CFT à la Segal [13], and similar to the vertex operator algebra framework for CFT initiated by Friedan & Shenker [14]. Indeed, in order to fully understand the universal nature of the conformal restriction property and the associated anomaly (e.g., as a characterizing property of canonical MKS loop measures, or as an inherent feature of any conformal field theory), it seems necessary to investigate the relationship of MKS measures to the geometric content of CFT.

Consider the conformal anomaly concretely with both $U \subset D$ simply connected. If the Brownian loop measure were finite, the anomaly could also be written as

$$\exp\left(\tfrac{1}{2}c(\kappa)\, m_D(\gamma, D \setminus U)\right) = \text{“}\, \frac{\exp\left(\tfrac{1}{2}c(\kappa)\, m_{\mathbb{D}}(\varphi_D(\gamma))\right)}{\exp\left(\tfrac{1}{2}c(\kappa)\, m_{\mathbb{D}}(\varphi_U(\gamma))\right)}\, \text{”},$$

comparing the mass $m_{\mathbb{D}}$ of Brownian loops in the unit disk $\mathbb{D}$ intersecting $\varphi_D(\gamma)$ and those intersecting $\varphi_U(\gamma)$, where $\varphi_D \colon D \to \mathbb{D}$ (resp. φ_U) is a uniformizing map from D (resp. U) onto the disk. One would be thus led to considering the SLE(κ) curve γ on two conformal structures (Riemannian metrics) on the disk induced by the maps φ_D and φ_U. To make this more precise, one can consider the metric dependence of *CFT partition functions* $Z_g(\Sigma)$ on surfaces (Σ, g), also related to total masses of SLE (or MKS) measures (cf. [10, 15, 16]). Namely, two metrics g and $e^{2\sigma}g$ in the same conformal class are related by the anomaly functional

$$S_L^0(\sigma, g) := \frac{1}{12\pi} \iint_\Sigma \left(\frac{1}{2}|\nabla_g \sigma|_g^2 + R_g \sigma\right) \mathrm{vol}_g + \frac{1}{12\pi} \int_{\partial\Sigma} k_g \sigma\, \widetilde{\mathrm{vol}}_g, \quad \sigma \in C^\infty(\Sigma, \mathbb{R}),$$

where ∇_g, R_g, vol_g, k_g, $\widetilde{\mathrm{vol}}_g$ are respectively the divergence, Gaussian curvature, and volume form on Σ, and the boundary curvature and volume form on $\partial\Sigma$, induced by g. Changes of metrics are thus encoded into an exponential factor $\exp(c\, S_L^0(\sigma, g))$, where $c \in \mathbb{R}$ is the *central charge*. Any CFT partition function transforms as $Z_{e^{2\sigma}g}(\Sigma) = e^{c\, S_L^0(\sigma,g)} Z_g(\Sigma)$. Dubédat formalized [15] how a comparison of partition functions (e.g., determinants of Laplacians) gives the conformal restriction anomaly (note though that partition functions might not be finite). E.g., taking $\Sigma = \mathbb{D}$ with flat metric g_0 and $U \subset \mathbb{D}$ simply connected, we expect

$$\frac{Z_{g_0}(\mathbb{D})}{Z_{|(\varphi_U^{-1})'|^2 g_0}(\mathbb{D})} \frac{Z_{|(\varphi_U^{-1})'|^2 g_0}(\mathbb{D} \setminus \varphi_U^{-1}(\gamma))}{Z_{g_0}(\mathbb{D} \setminus \gamma)} = \exp\left(\tfrac{c}{2}\, m_{\mathbb{D}}(\gamma, \mathbb{D} \setminus U)\right)$$

for $\gamma \subset U$ a chord (or Jordan loop; in which case the identity should hold up to a multiplicative factor that only depends on the conformal moduli of $\mathbb{D} \setminus \gamma$, $U \setminus \gamma$).

The aim in [12] is to explicitly derive the *Virasoro algebra* — the Lie algebra of infinitesimal conformal symmetries — from complex deformations $\mathrm{Def}_{\mathbb{C}}(S^1)$ associated to Jordan loops on surfaces, parameterized by the circle S^1. Morally, $\mathrm{Def}_{\mathbb{C}}(S^1)$ should be thought of as a complexification of the infinite-dimensional Fréchet-Lie group $\mathrm{Diff}_+^{\mathrm{an}}(S^1)$ (comprising real-analytic, orientation-preserving diffeomorphisms of S^1), whose Lie algebra $\mathfrak{X}_{\mathbb{R}}^{\mathrm{an}}(S^1)$ consists of real-analytic vector fields on S^1. Its complexification $\mathfrak{X}^{\mathrm{an}}_{\mathbb{C}}(S^1) := \mathfrak{X}^{\mathrm{an}}_{\mathbb{R}}(S^1) \otimes \mathbb{C}$ is known as the *Witt algebra*. It can be thought of as the Lie algebra of the complex deformations $\mathrm{Def}_{\mathbb{C}}(S^1)$, in the sense that flows of complex vector fields yield complex deformations. One is thus led to considering a moduli space of Riemann surfaces Σ with (analytically parametrized) boundary components (i.e., loops parameterized by the circle), endowed with a *sewing operation* across boundary components (viz. conformal welding), so boundary components become Jordan loops on the sewn surface.

In particular, the group $\mathrm{Diff}^{\mathrm{an}}_{+}(S^1)$ acts very naturally by reparameterization of the boundary components of Σ. However, in order to see the Virasoro structure which unifies the MKS loop measures, one defines a *real determinant line bundle* $\mathrm{Det}^c_{\mathbb{R}}$ on the moduli space, which is just a collection of real one-dimensional vector spaces associated to each equivalence class of surfaces Σ (encoding the conformal anomaly between metrics). The determinant line over Σ is defined as the collection of (formal) multiples of metrics under the equivalence $[e^{2\sigma} g] = e^{c S^0_L(\sigma,g)}[g]$. (By choosing a section for the determinant line bundle, the conformal anomaly $S^0_L(\sigma, g)$ can be written in a number of related but inequivalent ways [12], involving Loewner energy [17], Brownian loop measure [4, 11], or determinants of Laplacians [15].)

The moduli space carries an action of the complex deformations on the boundary components that truly change the complex structure of Σ, unlike mere reparameterizations. The (nontrivial) sewing operation on the moduli space and on the determinant line bundle then gives rise to an associative product for the associated central extension of $\mathrm{Def}_{\mathbb{C}}(S^1)$ by the multiplicative group of positive reals,

$$(\phi, \lambda) \cdot (\psi, \lambda') = (\phi\psi, \lambda\lambda' \, \Gamma_c(\phi, \psi)), \quad \phi, \psi \in \mathrm{Def}_{\mathbb{C}}(S^1),\ \lambda, \lambda' > 0,$$

where Γ_c is a *cocycle* describing algebraically the relevant central extension, and geometrically the nontrivial twist in the sewing operation. (Note, however, that complex deformations cannot form a well-defined Lie group, so one cannot naively speak of their Lie algebra, nor of their central extensions.) From this structure, one can explicitly compute the conformal anomaly in terms of the (honest) Lie algebra cocycle by taking two flows $(\phi_t)_{t\in\mathbb{R}}$ and $(\psi_t)_{t\in\mathbb{R}}$ of complex deformations generated by two vector fields $v, w \in \mathfrak{X}^{\mathrm{an}}_{\mathbb{C}}(S^1)$ in the Witt algebra [12]:

$$\frac{1}{2}\frac{\partial^2}{\partial t \partial s}\Big(\log \Gamma_c(\phi_t, \psi_s) - \log \Gamma_c(\psi_s, \phi_t)\Big)\Big|_{t=s=0} = \frac{c}{24\pi}\mathrm{Im}\int_0^{2\pi} v'(\theta) w''(\theta)\, \mathrm{d}\theta.$$

In particular, we see that the cocycle is nontrivial whenever the central charge $c \neq 0$ is nonzero, and that it coincides with the imaginary part of the celebrated *Gel'fand-Fuks* (or Virasoro) *cocycle*. Note that while the determinant line bundle is topologically trivializable, the sewing operation is not. This is the key fact that gives rise to a *nontrivial central extension* of $\mathrm{Def}_{\mathbb{C}}(S^1)$ and its Lie algebra, yielding the Virasoro structure and the nontrivial conformal anomaly. However, perhaps surprisingly the cocycle vanishes for real vector fields $v, w \in \mathfrak{X}^{\mathrm{an}}_{\mathbb{R}}(S^1)$, which shows that the complex deformations are necessary in order to see the conformal anomaly.

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