Towards a conformal field theory for Schramm-Loewner evolutions?

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Many models in statistical physics can be conveniently described in terms of their *geometric* features, such as *clusters* and *interfaces* (see Fig. 1). Of particular physical interest are so-called critical models, which exhibit fractal properties and (approximate) scale-invariance. For instance, critical Ising model is a spin system on a lattice describing magnetic material whose atoms form a regular crystalline structure. In the planar (2D) case, upon taking the mesh size of the lattice to zero (this is called the *scaling limit*), the model enjoys a strong symmetry, *conformal invariance*. To phrase the conformal invariance mathematically precisely, however, much care is needed. To this end, one possible approach is to describe scaling limits of the clusters and interfaces in terms of random geometric objects.

About 20 years ago, Oded Schramm made a breakthrough pertaining to the description of such interfaces: he introduced "stochastic Loewner evolution", now known as *Schramm-Loewner evolution* (SLE) curves. For a number of models, it has now been proven using (quite specific but very celebrated) discrete complex analysis methods (cf. [1, 2, 3] and references therein) that scaling limits of critical interfaces are indeed described by Schramm's SLEs. Alas, it seems beyond reach to prove by these techniques alone conformal invariance results for more general models, such as Potts, $O(n)$, or random-cluster models (see [4] for recent progress establishing rotational invariance, using still model-specific but quite general tools, but falling short of obtaining scale-invariance and thus conformal invariance).

Schramm argued that SLEs were the only possible random curves that could describe scaling limits of critical interfaces [3]. Indeed, upon requiring *conformal invariance in law* together with a *Markov property* for the growth of the curve, he observed that there is only a one-parameter family of possible random curves in the plane, SLE_κ. The parameter $\kappa > 0$ describes, in particular, fractal properties of SLE_κ curves. Soon after Schramm's breakthrough, John Cardy predicted a relationship between SLE curves and certain "boundary condition changing operators" in critical models [5]. Intuitively, the emergence of an interface from a boundary point should be governed by a "conformal field" $\Phi_{1,2}$, that is, an element in a *conformal field theory* (CFT) describing the scaling limit of the critical model.

Concretely, SLE_κ curves can be generated as *random Loewner evolutions* [7]: the time-evolution of the curve is encoded in a solution of Loewner's differential

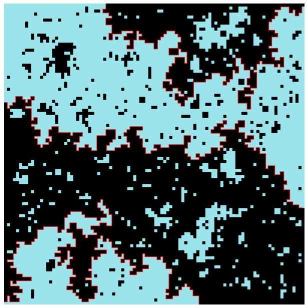

FIGURE 1. Configuration of critical Ising model with alternating boundary conditions (i.e., given boundary segments carry spins $+1$ and the other boundary segments spins -1). The macroscopic random interfaces connecting those boundary points where the boundary condition changes are highlighted. (Figure from [6].)

equation, which in the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ reads

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z, \qquad z \in \overline{\mathbb{H}},$$

where $t \mapsto W_t$ is a real-valued continuous function, *driving function*. The curve is obtained as $\gamma(t) := \lim_{\varepsilon \downarrow 0} g_t^{-1}(W_t + \mathrm{i}\varepsilon)$. For the simplest SLE_κ variant, *chordal* SLE, one takes $W = \sqrt{\kappa}B$, one-dimensional Brownian motion of speed κ. In general, interfaces in critical models with boundary conditions alternating at $2N$ marked boundary points (see Fig. 1) correspond to a collection $(\gamma_1, \ldots, \gamma_N)$ of N interacting random curves (e.g. [8, 9, 10, 11, 12, 13, 14, 15]). For $\kappa \leq 4$, one can uniquely characterize these curve families in terms of their conditional laws [11]: by a Markov chain coupling argument, there exists a unique probability measure on N interacting random curves $(\gamma_1, \ldots, \gamma_N)$ such that, for each j, the conditional law of γ_j given $\{\gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_N\}$ is chordal SLE_κ. Such a measure can also be constructed by weighting the product measure of N independent SLE_κ curves by a given Radon-Nikodym derivative (cf. [16, 9]). In light of Girsanov's theorem, the marginal law of γ_j in a multiple SLE_κ process in $(\mathbb{H}; x_1 < \cdots < x_{2N})$ is obtained by taking the driving function $W^{(j)}$ of γ_j to be the solution to SDE

$$\mathrm{d}W_t^{(j)} = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial_j \log \mathcal{Z}\big(g_t(x_1), \ldots, g_t(x_{j-1}), W_t, g_t(x_{j+1}), \ldots, g_t(x_{2N})\big)\,\mathrm{d}t,$$

with initial value $W_0^{(j)} = x_j$, where $\mathcal{Z}$ is a so-called *partition function* and $g_t(x_i)$ time-evolutions of the other marked points [17]. In the physics parlance [5, 18], the partition functions $\mathcal{Z}$ are examples of CFT correlation functions. While the concept of a "conformal field" is not well-defined in general, one can consider the correlation functions as encoding physical information. In some cases, it is known that a conformal field with given correlation functions can also be constructed as a random generalized function [19, 20]. One might then ask: *Is it possible to construct the appropriate SLE-CFT fields as random distributions?*

To this end, one could try to construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables $\Phi_{1,2}$ taking values in a suitable space $S'(\mathbb{R})$ of tempered distributions acting on test functions $f \in S(\mathbb{R})$, such that the moments

$$\mathbb{E}\big[(\Phi_{1,2}(f))^n\big] := \int_{\mathbb{R}^n} f(x_1) \cdots f(x_n) \; \mathcal{Z}(x_1, \ldots, x_n) \, \mathrm{d}x_1 \cdots \mathrm{d}x_n, \qquad n \in \mathbb{Z}_{>0},$$

where the kernels are given by suitable choices of SLE partition functions $\mathcal{Z}$ (here, vanishing when n is odd), would determine the law of $\Phi_{1,2}$ uniquely. One can check that Carleman's condition holds at least when $\kappa \in [2, 6]$ (thus showing uniqueness), while the construction is – to my knowledge – only established in the case where $\kappa = 4$ (using the Gaussian free field; work with Simon Schwarz & in progress).

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